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Relaxation of spherical systems with long-range interactions: a numerical investigation

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The process of relaxation of a system of particles interacting with long-range forces is relevant to many areas of Physics. For obvious reasons, in Stellar Dynamics much attention has been paid to the case of r^{-2} force law. However, recently the interest in alternative gravities emerged, and significant differences with respect to Newtonian gravity have been found in relaxation phenomena. Here we begin to explore this matter further, by using a numerical model of spherical shells interacting with an $r^{-\alpha}$ force law obeying the superposition principle. We find that the virialization and phase-mixing times depend on the exponent α, with small values of α corresponding to longer relaxation times, similarly to what happens when comparing for N-body simulations in classical gravity and in Modified Newtonian Dynamics.

Keywords: Relaxation, Galactic dynamics, MOND, Long-range interactions.

1. Introduction

Phase Mixing and Violent Relaxation are important dynamical phenomena, relevant in the astrophysical contest ([Lynden-Bell, 1967], [Binney & Tremaine, 2008]). In fact N-body simulations revealed that the final states of gravitating systems experiencing a collapse from *cold* and *clumpy* initial conditions are structurally very similar to real elliptical galaxies ([van Albada, 1982], [Bertin *et al.*, 2005], [Nipoti *et al.*, 2006]). In addition, the phase-space properties of the numerical end-products are remarkably interesting, with a differential energy distribution described very well by an exponential law, over a large value of energy values ([Binney, 1982], [Bertin & Stiavelli, 1984], [Ciotti, 1991]). Finally, recent investigations of Modified Newtonian Dynamics (MOND) showed that relaxation processes are much slower in MOND than in Newtonian gravity ([Ciotti *et al.*, 2007], [Nipoti *et al.*, 2007]). This fact is relevant here, because in the weak limit MOND forces behave qualitatively as r^{-1}. Unfortunately, MOND is a non-linear theory, so that numerical simulations are more difficult than for systems with forces obeying the superposition principle. With this paper we begin an exploration aimed at understanding how the properties mentioned above depend on the specific nature of long-range (power-law) interactions. In particular, we study the relaxation of systems governed by additive $r^{-\alpha}$ forces, focusing on the simplified case of spherical systems.

2. The model

Following previous studies ([Hénon, 1962], [Takizawa & Inagaki, 1997], [Sanders, 2008], [Malekjiani *et al.*, 2009]), we numerically integrate the motion of $N = 10^4$ massive spherical shells. We assume that a surface element of each shell, of mass δm and at vector position $\mathbf{y}$, produces an acceleration at the point $\mathbf{x}$ given by

$$\mathbf{g} = -G\delta m \frac{\mathbf{x}-\mathbf{y}}{||\mathbf{x}-\mathbf{y}||^{\alpha+1}}, \tag{1}$$

where G is the "gravitational" constant, and the force index α, for reasons of convergence, is restricted to $\alpha < 3$; the case of Newtonian gravity is obtained for $\alpha = 2$, while $\alpha = -1$ corresponds to the case of harmonic force. The equations of motion for the shell i, of mass m_i and radius r_i, are

$$\frac{dr_i}{dt} = v_i, \quad \frac{dv_i}{dt} = a_i \equiv \frac{J_i^2}{r_i^3} + g_{ii} + \sum_{j\neq i=1}^{N} g_{ji}, \tag{2}$$

where v_i and J_i are the radial velocity and the (constant) angular momentum per unit mass of the shell, and a_i is its effective radial acceleration. The field per unit mass acting on each surface element of the shell i due to the shell j at $r_j \neq r_i$ is

$$g_{ji} = -\frac{Gm_j}{4r_i^2 r_j} \times \begin{cases} \dfrac{(r_i+r_j)^{3-\alpha} - |r_i-r_j|^{3-\alpha}}{3-\alpha} - (r_i^2 - r_j^2)\dfrac{(r_i+r_j)^{1-\alpha} - |r_i-r_j|^{1-\alpha}}{\alpha-1}, & (\alpha \neq 1) \\ 2r_i r_j - (r_i^2 - r_j^2)\ln\dfrac{|r_i-r_j|}{r_i+r_j}, & (\alpha = 1) \end{cases} \tag{3}$$

while

$$g_{ii} = -\frac{2^{1-\alpha}}{3-\alpha}\frac{Gm_i}{r_i^\alpha} \tag{4}$$

is the self-field per unit mass acting on each surface element of the shell i; for $\alpha = 2$ the celebrated Newton theorems are recovered. Note that g_{ii} can be obtained from g_{ji} by imposing $m_j = m_i$ and taking the limit $r_j \to r_i$; the same limit is required to obtain the force in case of shell superposition.

We performed experiments with different values of the exponent α, considering both radial collapses ($J_i = 0$) and collapses with $J_i \neq 0$. Numerically, when a shell collapses to the origin, the sign of the velocity v_i reverses to positive. As indicators of dynamical relaxation we consider the virial ratio $2K/|W|$, the differential energy distribution $n(E)$, and the phase-space evolution of the pairs (r_i, v_i). We evolved the systems up to 50 dynamical times ($t_{\rm dyn}$), where on dimensional grounds

$$t_{\rm dyn} \equiv \sqrt{\frac{2r_0^{\alpha+1}}{GM_{\rm tot}}}; \tag{5}$$

$M_{\rm tot} = \sum_i m_i$ is the total mass of the system, and r_0 is the half-mass radius at $t = 0$. We explored different radial distributions for the initial density profile, noticing that the final products are not strongly dependent on it. The initial conditions are characterized by a vanishing virial ratio, typical of cold collapses. The integration scheme ([Di Cintio, 2009]) uses a standard 2^{th} order leapfrog algorithm with constant timestep (e.g., see [Grubmüller *et al.*, 1991]).

3. The virial ratio

We first consider the time evolution of the virial ratio $2K/|W|$. As the density distribution of the system can be written as

$$\rho(r) = \sum_{i=1}^{N} \frac{m_i}{4\pi r_i^2}\delta(r - r_i), \tag{6}$$

the virial function W [1] ([Binney & Tremaine, 2008]) reduces to

$$W = 4\pi \int_0^\infty \rho(r)\, g(r)\, r^3\, dr = \sum_{i=1}^{N} m_i r_i \left(g_{ii} + \sum_{j\neq i=1}^{N} g_{ji} \right), \tag{7}$$

while the total kinetic energy of the system is

$$K = \sum_{i=1}^{N} \frac{m_i}{2} \left(v_i^2 + \frac{J_i^2}{r_i^2} \right). \tag{8}$$

As apparent from Fig. 1, in purely radial collapses (and in collapses with $J_i \neq 0$, not shown), the evolution

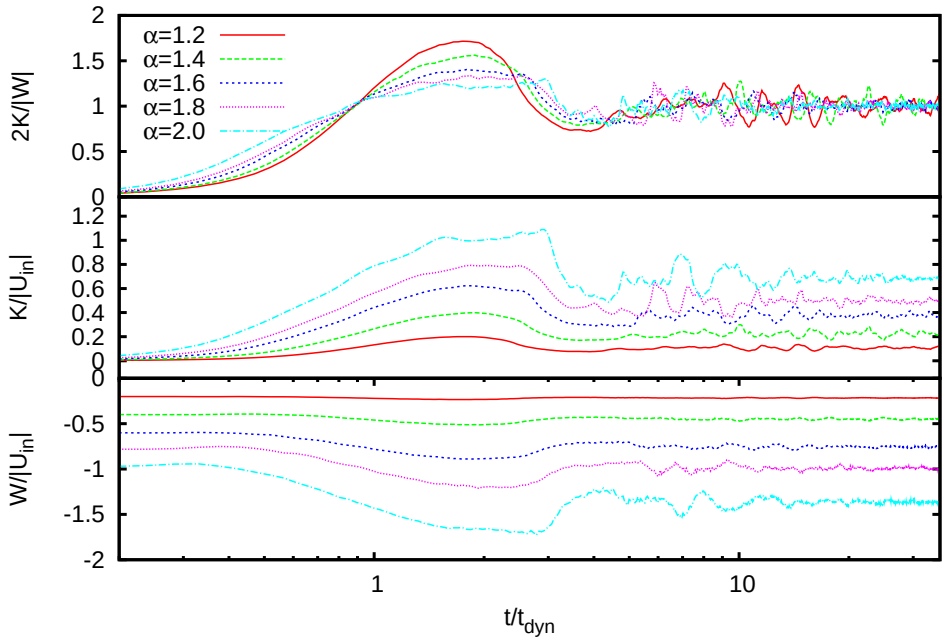

Fig. 1. Top panel: evolution of the virial ratio over $\sim 36 t_{\rm dyn}$ for systems with $J_i = 0$, and different values of the force index α (only bound shells have been considered in the plot). Low values of α correspond to larger and more long-lasting virial oscillations. Middle and bottom panel: kinetic energy and virial W in units of the initial binding energy $U_{\rm in}$. Virial theorem and energy conservation show that for cold collapses in absence of escapers, $W = 2(\alpha-1)U_{\rm in}/(3-\alpha)$ at virialization.

of the virial ratio shows a violent jump followed by an oscillatory phase, around the equilibrium value 1. The violent phase lasts $\simeq 2t_{\rm dyn}$, while the subsequent virial oscillations last considerably longer. A certain trend with α is apparent: as the exponent α increases towards 2 (the force range shortens), the peak of the first and of the successive jumps lowers. Remarkably, for the lowest value of α explored, (i.e., for the forces with the slowest decline with distance), the virial oscillations continue for several tens of dynamical times, similarly to what found in numerical simulations in MOND collapses. We interpret this result as a clear trend of the system behavior towards the harmonic oscillator case $\alpha = -1$, when from eqs. (3) and (4) it follows that $a_i = -GM_{\rm tot} r_i + J_i^2/r_i^3$, so that each shell oscillates independently of the others and no energy exchanges between shells can take place. It is important to note that the somewhat arbitrary definition of $t_{\rm dyn}$ in eq. (5) is found to be a quite accurate measure of the true dynamical time, as the first peak of the virial ratio occur at $t \simeq 2t_{\rm dyn}$ for all the explored values of α.

[1] For potentials that are homogenous functions of r, as here for $\alpha \neq 1$, $W = (\alpha-1)U$, where U is the potential energy. For $\alpha = 1$ instead $W = -GM_{\rm tot}^2/2$ is constant, as for deep-MOND forces [Nipoti *et al.*, 2007].

4. Phase-space evolution

The same behavior is also shown by phase-space evolution of the pairs (r_i, v_i), represented in Fig. 2 for systems with $\alpha = 1.2$ and $\alpha = 2$. In the former case (top panels), the phase-space still presents an ordered structure after $\simeq 10t_{\rm dyn}$, symptom of a less efficient mixing than in the Newtonian case (bottom panels). This is reminiscent of the behavior of MOND systems, and the plots are strikingly similar to those in Fig. 2 in [Ciotti *et al.*, 2007]. This finding is also confirmed by experiments with $\alpha \leq 1$ and $\alpha > 2$ (not shown). From Fig. 2 (bottom panels) is apparent how shells are lost from the system with $\alpha = 2$; note that ejection is energetically impossible when $\alpha \leq 1$.

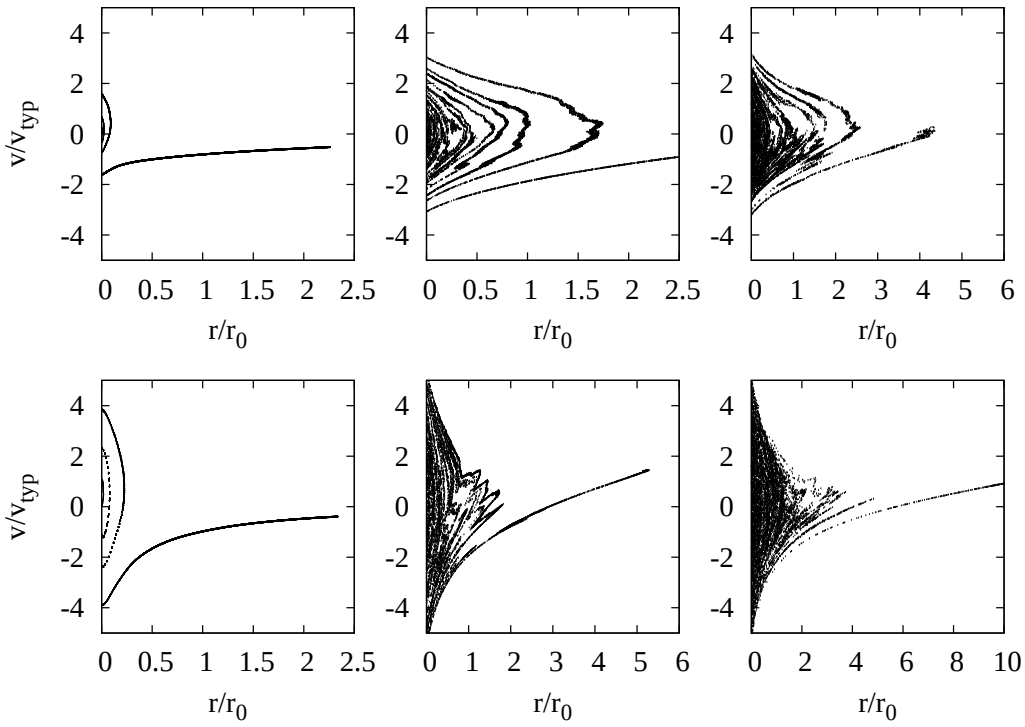

Fig. 2. From left to the right: phase-space at $0.7t_{\rm dyn}$, $7t_{\rm dyn}$, and $14t_{\rm dyn}$, for a system with $\alpha = 1.2$ (top panels) and $\alpha = 2$ (bottom panels). The scale velocity is $v_{\rm typ} \equiv r_0/t_{\rm dyn}$, while r_0 and $t_{\rm dyn}$ are defined in Sect. 2. Note how phase mixing and shell ejection are more efficient in the case of Newtonian forces.

5. Differential energy distribution

We finally present the differential energy distribution at virialization, $n(E)$. The quantity $n(E)dE$ is defined as the number of shells with energy in the range $(E, E + dE)$. The total energy of each shell is

$$E_i = K_i + m_i \left(\frac{\phi_{ii}}{2} + \sum_{j \neq i = 1}^{N} \phi_{ji} \right), \tag{9}$$

where an integration gives the expressions for the potential ϕ_{ji} at r_i due to the shell j, and the self-potential[2] ϕ_{ii}. For $\alpha \neq 1$

$$\phi_{ji} = -\frac{Gm_j}{2r_i r_j} \frac{(r_i + r_j)^{3-\alpha} - |r_i - r_j|^{3-\alpha}}{(\alpha - 1)(3 - \alpha)}; \quad \phi_{ii} = -\frac{Gm_i 2^{2-\alpha}}{(\alpha - 1)(3 - \alpha) r_i^{\alpha - 1}}, \tag{10}$$

[2]Note that $-d\phi_{ji}/dr_i = g_{ji}$, while $-d\phi_{ii}/dr_i = 2g_{ii}$. Note also that $\sum E_i$ is not the total energy of the system.

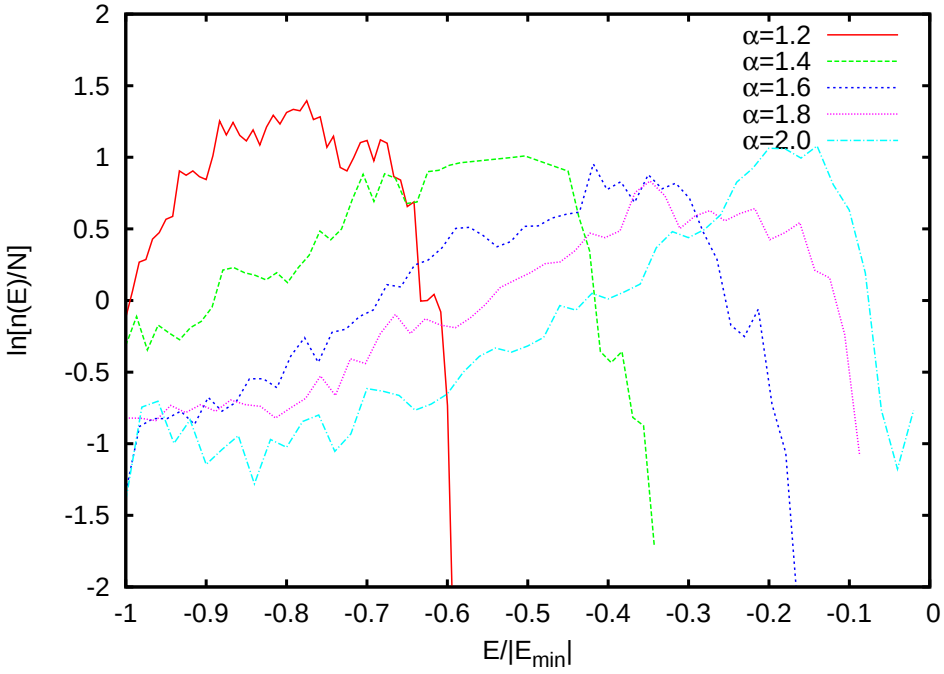

Fig. 3. Final differential energy distribution, normalized to $E_{\rm min}/N$, as a function of α. $E_{\rm min}$ is the minimum energy value attained by the shell distribution.

while for $\alpha = 1$

$$\phi_{ji} = \frac{Gm_j}{4r_i r_j}[(r_i + r_j)^2 \ln(r_i + r_j) - (r_i - r_j)^2 \ln|r_i - r_j| - 2r_i r_j], \quad \phi_{ii} = Gm_i \left(\ln r_i + \ln 2 - \frac{1}{2}\right). \quad (11)$$

The formulae for the self-potential can be obtained by direct integration or by taking $m_j = m_i$ and $r_j \to r_i$ in the expression of ϕ_{ji}. The final $n(E)$ for systems with $1.2 < \alpha < 2$ is shown in Fig. 3. The distributions (albeit quite noisy, due to the limited number of shells used in the simulations) roughly reproduce the expected decline with increasing (absolute) energy predicted by an exponential function

$$n(E) = n_0 e^{-\beta|E|}, \quad \beta > 0, \quad (12)$$

([Binney, 1982], [Ciotti, 1991]). However, the agreement of our results with eq. (12), known to provide an excellent fit over a large energy range of the virialized end-products of N-body simulations of *cold* and *clumpy* systems, is quite poor. This is not surprising, as the imposed spherical symmetry of the shells does not allow the additional mixing effects due to non-radial density inhomogeneities. We conclude by noticing that the case $\alpha = 1.2$ presents a curious, non-monothonic trend with energy; we verified that this feature is not a numerical artifact. We stress that also in MOND numerical simulations (that should correspond, in a broad sense, to the low-α cases) the final $n(E)$ is very different to that of Newtonian simulations starting from the same initial conditions (see Fig. 5 in Nipoti *et al.*, 2007).

6. Discussion and conclusions

With the aid of a simplified model of N massive, spherically symmetric and concentric shells, moving under the action of long-range, power-law additive forces, we studied the relaxation process leading to the final virialized states. We find that for all the explored values of α the relaxation process involves first a rapid Violent Relaxation phase, followed by a longer and gentle relaxation due to Phase Mixing. For α increasing from values < 1 to > 2, the relaxation time (in units of dynamical time) decreases steadily. In general, systems involving force laws with low α have longer mixing times compared to those of systems with high α, that are by contrast characterized by faster evaporation, resulting in a well marked core-halo

segregation. This finding can be useful to improve our understanding of MOND, where the relaxation mechanisms (due to the nonlinearity of the theory) are still quite poorly understood; in fact, the cases of low α present an evolution similar to that of N-body MOND systems. Finally, the $n(E)$ distribution of the final virialized states are qualitatively similar (with one exception), increasing for increasing energies, but are not described particulary well by an exponential law (especially for low α). The next step in our study will be the investigation of the relaxation of N-body systems with interparticle additive forces as in eq. (1). One of the main reasons for the successive analysis is to abandon the assumption of spherical symmetry, as it is expected that the increasing number of degrees of freedom will produce a faster relaxation, and a better agreement with an exponential $n(E)$. Also, it will be interesting to investigate how radial orbit instability, characteristic of self-gravitating systems with most particles on radial orbits, depends on the specific nature of the force law. Unfortunately, for generic values of α, a field equation for the potential (such as Poisson equation for Newtonian gravity, or the p-Laplace equation for MOND systems) does not exists, so that particle-mesh methods cannot be used, and only the more time-expensive direct N-body numerical schemes are available. An original code is currently under testing.

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